International Journal of Theoretical Physics, Vol. 37, No. 11, 1998

Wheeler Propagator

C. G. Bollini[1] and M. C. Rocca[1,2]

Received February 19, 1998

We study the half-advanced and half-retarded Wheeler Green function and its relation to Feynman propagators, first for massless equation, then, for Klein–Gordon equations with arbitrary mass parameters, real, imaginary, or complex. In all cases the Wheeler propagator lacks an on-shell free propagation. The Wheeler function has support inside the light-cone (whatever the mass). The associated vacuum is symmetric with respect to annihilation and creation operators. We show with some examples that perturbative unitarity holds, whatever the mass (real or complex). Some possible applications are discussed.

1. INTRODUCTION

More than half a century ago, Wheeler and Feynman (1945) represented electromagnetic interactions by means of a half advanced and half retarded potential. The charged medium was suppossed to be a perfect absorber, so that no radiation could possibly escape the system. We are going to call this kind of potential a "Wheeler function" (or propagator), although it had been used before by Dirac (1938) when trying to avoid some runaway solutions. Later, Wheeler and Feynman (1949) showed that, in spite of the fact that it contains an advanced part, the results do not contradict causality.

Of course, the success of QED and renormalization theory soon made it unnecesary or not advisable to follow that line of research (at least for electromagnetism).

One of the distinctive characteristics of the Green function used in Wheeler and Feynman (1945, 1949) and Dirac (1938) is its lack of asymptotic free waves. This is the reason behind the choice of a "perfect absorber" for

[1] Departamento de Física, Fac. de Ciencias Exactas, Universidad Nacional de La Plata, C.C. 67 (1900) La Plata, Argentina.
[2] Departamento de Matemáticas, Fac. de Ciencias Exactas, Universidad Nacional del Centro de la Pcia de Bs. As., Pintos 390, C.P. 7000, Tandil, Argentina.

0020-7748/98/1100-287 7$15.00/0

the medium through which the field propagates. As the quantization of free waves is associated with free particles, the above-mentioned feature of Wheeler functions implies that no free quantum of the field can ever be observed. Nevertheless, we are now used to the existence of confined particles. They do not manifest themselves as free entities. We can give some examples (outside QCD) where such a behavior can be present.

A Lorentz-invariant higher order equation can be decomposed into Klein–Gordon factors, but the corresponding mass parameters need not be real. For instance, the equation

$$(\Box^2 + m^4)\varphi = (\Box + im^2)(\Box - im^2)\varphi = 0 \tag{1}$$

gives rise to a pair of constituent fields (Barci *et al.*, 1994a) obeying

$$(\Box \pm im^2)\varphi_\pm = 0 \tag{2}$$

Any solution of (2) blows up asymptotically. We can say that the corresponding fields should be forbidden to appear asymptotically as free waves. Therefore, they should have a Wheeler function as propagator (Bollini and Oxman, 1992).

Equations similar to (1), or more generally

$$(\Box^n \pm m^{2n})\varphi = 0 \tag{3}$$

appear in a natural way in supersymmetric models for higher dimensional spaces (Bollini and Giambiagi, 1985).

Another example is provided by fields obeying Klein–Gordon equations with the wrong sign of the mass term. A careful analysis shows that the propagator should be a Wheeler function (Barci *et al.*, 1993, 1994b). Accordingly, no tachyon can ever be observed as a free particle. They can only exist as "mediators" of interactions.

To define the propagators in a proper way, we have to solve the equations for the Green functions, with suitable boundary conditions.

For the wave equation

$$\Box \tilde{G}(x) = \delta(x) \tag{4}$$

a Fourier transformation gives

$$G(p) = (\bar{p}^2 - p_0^2)^{-1} \equiv (p_\mu p^\mu)^{-1} \equiv P^{-1} \tag{5}$$

Of course, it is necessary to specify the nature of the singularity. Different determinations imply different types of Green functions. For the classical solution of (4) it is natural to use the retarded function ($\tilde{G}_{rt}$). It corresponds to the propagation toward the future of the effect produced by the sources. This function can be obtained by means of a Fourier transform of (5) in which the p_0 integration is taken along a path from $-\infty$ to $+\infty$, leaving the poles to the right. In practice, we add to p_0 a small, positive imaginary part:

$$G_{rt}(p) = [\bar{p}^2 - (p_0 + i0)^2]^{-1}$$
$$(\bar{p}^2 - p_0^2 - i0 \text{ sgn } p_0)^{-1} = (P - i0 \text{ sgn } p_0)^{-1} \tag{6}$$

The advanced solution is the complex conjugate of (6):

$$G_{ad}(p) = (\bar{p}^2 - p_0^2 + i0 \text{ sgn } p_0)^{-1} = (P + i0 \text{ sgn } p_0)^{-1} \tag{7}$$

For the Feynman propagator we have to add a small imaginary part to P (not just to p_0):

$$G_{\pm}(p) = (P \pm i0)^{-1} \tag{8}$$

and in the massive case

$$G_{\pm}(p) = (P + m^2 \pm i0)^{-1} \tag{9}$$

The Wheeler function is half advanced and half retarded. It is easy to see that it is also half Feynman and half its conjugate (we will not use any index for the Wheeler propagator):

$$G(p) = \frac{1}{2} G_{+}(p) + \frac{1}{2} G_{-}(p) \tag{10}$$

On the real axis, the Wheeler function coincides with Cauchy's "principal value" Green function, which is known to be zero on the mass-shell (no free waves).

To perform convolution integrations in p space, we will utilize the method presented in Bollini and Giambiagi (1996). Essentially, it consists in the use of the Bochner theorem for the reduction of the Fourier transform to a Hankel transform. The nucleus of this transformation is made to correspond to an arbitrary number of dimensions ν, taken as a free parameter. In this way, starting with a given propagator in p space, we get a function in x space whose singularity at the origin depends analytically on ν. There is then a range of values (of ν) such that the product of Green functions is allowed and well determined.

In general, for a function $f(P \pm i0)$ we have (Bollini and Giambiagi, 1996; Bochner, 1939):

$$\mathcal{F}\{f(P \pm i0)\}\ (x) = \mp \frac{i}{x^{\nu/2}} \int_0^\infty dy\, y^{\nu/2} f(y^2)\, \mathcal{J}_{\nu/2-1}(xy) \tag{11}$$

where

$$x = (Q \mp i0)^{1/2}$$
$$Q = r^2 - x_0^2 = x_\mu x^\mu$$

The r.h.s. of (11) is a Hankel transform of the function $f(y^2)$ (Gradshteyn and Ryzik, 1980; Erdelyi, 1954).

2. WHEELER FUNCTIONS

As a first example we take the massless case, for which (11) and Gradshteyn and Ryzhik, (1980) and Guelfand and Shilov (1972) give

$$\mathcal{F}\{(P \pm i0)^\alpha\}\,(x) = \mp i2^{2\alpha+\nu/2} \frac{\Gamma(\alpha + \nu/2)}{\Gamma(-\alpha)} (Q \mp i0)^{-\alpha-(\nu/2)} \tag{12}$$

The massless Wheeler propagator is [cf. equation (10)]

$$P^\alpha = \frac{1}{2}(P + i0)^\alpha + \frac{1}{2}(P - i0)^\alpha \tag{13}$$

Its Fourier transform is then

$$\mathcal{F}\{P^\alpha\}(x) = i2^{2\alpha+(\nu/2)} \frac{\Gamma(\alpha + \nu/2)}{\Gamma(-\alpha)} \times \left[\frac{1}{2}(Q + i0)^{-\alpha-(\nu/2)} - \frac{1}{2}(Q - i0)^{-\alpha-(\nu/2)}\right] \tag{14}$$

But we also have the relation (Guelfand and Shilov, 1972)

$$(Q \pm i0)^\lambda = Q_+^\lambda + e^{\pm i\pi\lambda}\, Q_-^\lambda \tag{15}$$

where

$$Q_+^\lambda = \begin{cases} Q^\lambda & Q > 0 \\ 0 & Q \leq 0 \end{cases}$$

$$Q_-^\lambda = \begin{cases} (-Q)^\lambda & Q < 0 \\ 0 & Q \geq 0 \end{cases}$$

so that we can write (14) as

$$\mathcal{F}\{P^\alpha\}(x) = 2^{2\alpha+(\nu/2)} \frac{\Gamma(\alpha + \nu/2)}{\Gamma(-\alpha)} \sin\pi\left(\alpha + \frac{\nu}{2}\right) Q_-^{-\alpha-(\nu/2)} \tag{16}$$

Equation (16) shows another interesting property of Wheeler functions.

They are real and have support inside the light-cone of the coordinates. Furthermore, for $\alpha = -1$, the trigonometric function tends to zero for $\nu \to 4$, but $Q_-^{1-(\nu/2)}$ has a pole at $\nu = 4$ with residue $\delta(Q)$ (Faddeev and Slavnov, 1970). Then

$$\mathcal{F}\{P^{-1}\}(x) = \delta(Q) \tag{17}$$

In four dimensions the massless Wheeler function is concentrated on the light cone.

For the massive case the Wheeler propagator is

$$(P + m^2)^{-1} = \frac{1}{2}(P + m^2 + i0)^{-1} + \frac{1}{2}(P + m^2 - i0)^{-1} \tag{18}$$

The Fourier transform of the Feynman propagators is (Guelfand and Shilov, 1972)

$$\mathcal{F}\{(P + m^2 \pm i0)^{-1}\}\,(x) = \mp\, i m^{\nu/2-1}\,[Q_+^{1/2(1-\nu/2)}\,\mathcal{K}_{\nu/2-1}(mQ_+^{1/2}) + i\frac{\pi}{2}\,Q_-^{1/2(1-\nu/2)}\,\mathcal{H}^{\beta}_{1-\nu/2}\,(mQ_-^{1/2})] \tag{19}$$

where $\beta = 1$ for the upper sign and $\beta = 2$ for the lower sign.

Equations (18) and (19) give

$$\mathcal{F}\{(P + m^2)^{-1}\}(x) = \frac{\pi}{2}\, m^{\nu/2-1} Q_-^{1/2(1-\nu/2)}\, \mathcal{J}_{1-\nu/2}(mQ_-^{1/2}) \tag{20}$$

Also for the massive case, the Wheeler function is zero outside the light-cone. [For the definition of Bessel functions we follow Gradshteyn and Ryzhik (1980, p. 951, 8.40).]

We can evaluate convolutions by means of the well-known convolution theorem. The Fourier transform of a convolution is the product of the Fourier transforms of each factor:

$$f(p) * g(p) = c\mathcal{F}^{-1}\{\mathcal{F}\{f(p)\}(x)\mathcal{F}\{g(p)\}(x)\}(p)$$
$$c = (2\pi)^{\nu/2} \tag{21}$$

The product of distributions inside the curly brackets can be taken in a suitable range of ν and analytically extended to other values (Bollini and Giambiagi, 1996).

The convolution of two Feynman functions gives another Feynman function:

$$(P - i0)^{-1} \cdot (P - i0)^{-1} = 2ia(\nu)\,(P - i0)^{\nu/2-2} \tag{22}$$

where

$$a(\nu) = c2^{-\nu/2-1}\Gamma^2\left(\frac{\nu}{2}-1\right)\frac{\Gamma(2-\nu/2)}{\Gamma(\nu-2)}$$

and a similar equation holds with $i \to -i$.

By using (16) and (21) we get for the Wheeler propagator

$$P^{-1} * P^{-1} = a(\nu)\ tg\ \pi\left(\frac{\nu}{2}-1\right)P^{\nu/2-2} \tag{23}$$

where

$$a(\nu) = c2^{-\nu/2-1}\,\Gamma^2\left(\frac{\nu}{2}-1\right)\frac{\Gamma\,(2-\nu/2)}{\Gamma\,(\nu-2)}$$

Equation (22) shows a pole for $\nu \to 4$ (the usual ultraviolet divergence), while (23) is well determined in that limit (the self-energy for Wheeler propagator do not have ultraviolet divergence).

3. TACHYONS

A tachyon field obeys a Klein–Gordon equation with the wrong sign of the "mass" term. The Green function is an inverse of $P - \mu^2$ (we use $\mu^2 = -m^2$ for the "mass" of the tachyon). To find the corresponding Wheeler function we go back to the original definition, namely, a half retarded and half advanced propagator:

$$(P-\mu^2)^{-1} = \frac{1}{2}(P-\mu^2)^{-1}_{\text{ad}} + \frac{1}{2}(P-\mu^2)^{-1}_{\text{rt}} \tag{24}$$

The Fourier transform of the advanced part is

$$\mathscr{F}\{(P-\mu^2)^{-1}_{\text{ad}}\}(x) = \frac{1}{(2\pi)^{\nu/2}}\int d^{\nu-1}p\ e^{i\vec{p}\cdot\vec{r}}\int_{\text{ad}} dp_0\,\frac{e^{-ip_0x_0}}{\vec{p}^2 - p_0^2 - \mu^2}$$

where the path of integration runs parallel to the real axis and below both poles of the integrand. For $x_0 > 0$ the path can be closed on the lower half-plane of p_0 giving a null result. For $x_0 < 0$, on the other hand, we have the contribution of the residues at the poles:

$$p_0 = \pm\omega = \pm\sqrt{\bar{p}^2 - \mu^2} \qquad \text{if} \quad \bar{p}^2 \geq \mu^2$$
$$p_0 = \pm i\omega' = \pm i\sqrt{\mu^2 - \bar{p}^2} \qquad \text{if} \quad \bar{p}^2 \leq \mu^2$$

$$\mathcal{F}\{(P - \mu^2)_{\text{ad}}^{-1}\}(x) = \frac{-2\pi}{(2\pi)^{\nu/2}} \int d^{\nu-1}p\, e^{i\bar{p}\cdot\bar{r}} \left[\frac{\sin\omega x_0}{\omega}\,\Theta(\bar{p}^2 - \mu^2) + \frac{\text{sh}\,\omega' x_0}{\omega'}\,\Theta(\mu^2 - \bar{p}^2)\right]$$
$$= \frac{-1}{(2\pi)^{\nu/2}} \int d^{\nu-1}p\, e^{i\bar{p}\cdot\bar{r}}\,\frac{\sin\Omega\, x_0}{\Omega} \tag{25}$$

where $\Omega = (\bar{p}^2 - \mu^2 + i0)^{1/2}$ [cf. (15)] (Θ is Heaviside's function). Finally, using (11) and formula 6.737-5 (p. 761) of Gradshteyn and Ryzhik (1980), we obtain

$$\mathcal{F}\{(P - \mu^2)_{\text{ad}}^{-1}\}(x) = \pi\mu^{\nu/2-1}\, Q_-^{1/2(1-\nu/2)}\, \mathcal{J}_{1-\nu/2}(\mu Q_-^{1/2}) \tag{26}$$

For the retarded part we get a similar result, with the substitution $x_0 \to -x_0$.

Again, we see that the Wheeler propagator has support inside the light-cone, but instead of a Bessel function of the first kind, we have now a Bessel function of the second kind.

Note also that for the tachyon, the Wheeler function is not half Feynman and half its complex conjugate. This fact is due to the presence of the imaginary poles (at $p_0 = \pm i\omega'$).

4. FIELDS WITH COMPLEX MASS PARAMETERS

The decomposition in Klein–Gordon factors of a higher order equation often leads to complex mass parameters. Equation (1) is an example. The constituent fields obey (2). A simple higher order equation such as (3) presents the same behavior. Of course for a real equation the masses come in complex conjugate pairs. We consider

$$(\Box - M^2)\phi = 0, \qquad M = m + i\mu \qquad (m > 0) \tag{27}$$

This type of equation has been analyzed elsewhere (Bollini and Oxman, 1992). The Green functions are inverses of $P + M^2 = \Omega^2 - p_0^2$, where $\Omega = (\bar{p}^2 + M^2)^{1/2}$. When $\bar{p}^2$ varies from 0 to ∞ the two poles at $p_0 \pm \Omega$ move on a line contained in a horizontal band of width $\pm i\mu$, centered at the real axis.

The retarded Green function is obtained with a p_0 integration that runs parallel to the real axis, with Im $p_0 > |\mu|$. For the advanced solution, the integration runs below both poles (Im $p_0 < -|\mu|$).

With this procedure we get [compare with (26)]

$$\mathcal{F}\{(P + M^2)^{-1}\}(x) = \frac{\pi}{2} M^{\nu/2-1} Q_-^{1/2(1-\nu/2)} \mathcal{J}_{(\nu-3)/2}(MQ_-^{1/2}) \tag{28}$$

Now we have the general result: The Wheeler function propagates inside the light-cone for any value of the mass, real (bradyons), imaginary (tachyons), or complex ($M = m + i\mu$).

In the case of complex masses, a natural definition for the Feynman propagator is obtained by a p_0 integration along the real axis. Then it is not difficult to see that

$$\mathcal{F}\{(P + M^2)_F^{-1}\}(x) = \sqrt{\frac{\pi}{2}}\, r^{(3-\nu)/2} \int_0^\infty dk\, k^{(\nu-1)/2} \left(\frac{\sin \Omega|t|}{\Omega} - i \operatorname{sgn} \mu \frac{\cos \Omega|t|}{\Omega}\right) \mathcal{J}_{(\nu-3)/2}(rk) \tag{29}$$

The first term in the right-hand side is the Wheeler function. The second term corresponds to a positive loop around the pole in the upper half-plane and a negative loop around the pole in the lower half-plane.

If we say that the conjugate Feynman function (*not* the complex conjugate) is obtained by changing the signs of both loops, then the Wheeler function is also half Feynman and half its conjugate.

The term in cos $\Omega|t|$ can be read in Gradshteyn and Ryzhik (1980, 6.735-6).

5. ASSOCIATED VACUUM

It is well known that the perturbative solution to the quantum equation of motion leads to a Green function which is the vacuum expectation value of the chronological product of field operators (VEV). Furthermore, when the quanta are not allowed to have negative energies, the VEV turns out to be Feynman's propagator.

However, when the energy-momentum vector is spacelike the sign of its energy component is not Lorentz invariant. It is then natural to have symmetry between positive and negative energies. It has been shown that under this premise, the VEV is a Wheeler propagator (Barci *et al.*, 1993, 1994b).

To see clearly the origin of the difference between both types of propagators, we are going to compare the usual procedure with the symmetric one.

A quantized Klein–Gordon field can be written as

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega}} [a(\bar{k})e^{ik\cdot x} + a^+(\bar{k})e^{-ik\cdot x}] \tag{30}$$

where

$$[a(\bar{k}), a^+(\bar{k}')] = \delta(\bar{k} - \bar{k}'); \qquad \omega = \sqrt{\bar{k}^2 + m^2}$$

For simplicity, we are going to consider a single (discretized) degree of freedom.

The raising and lowering operators obey

$$[a, a^+] = 1 \tag{31}$$

The energy operator is

$$h = \frac{\omega}{2}(aa^+ + a^+a) = \omega a^+ a + \frac{\omega}{2} = h_0 + \frac{\omega}{2}$$

Usually, the energy is redefined to be h_0. The vacuum then obeys

$$h_0|0\rangle = 0 \tag{32}$$

It is a consequence of (31) and (32) that

$$\langle 0|aa^+|0\rangle = 1, \qquad \langle 0|a^+ a|0\rangle = 0 \tag{33}$$

On the other hand, the symmetric vacuum is defined to be the state that has zero "true energy":

$$h|0\rangle = \frac{\omega}{2}(aa^+ + a^+ a)\,|0\rangle = 0 \tag{34}$$

Equations (31) and (34) imply

$$\langle 0|aa^+|0\rangle = \frac{1}{2}, \qquad \langle 0|a^+ a|0\rangle = -\frac{1}{2} \tag{35}$$

Let as assume, for the sake of the argument, that we define a "ceiling" state (as opposed to a ground state):

$$a^+|0\rangle = 0 \tag{36}$$

Equations (31) and (36) give

$$\langle 0|aa^+|0\rangle = 0, \qquad \langle 0|a^+ a|0\rangle = -1 \tag{37}$$

The usual normal case, equation (33), leads to the Feynman propagator. The "inverted" case, equated (37), leads to its complex conjugate. Then (35), which is one-half of (33) and one-half of (37), leads to one-half of the Feynman function and one-half of its conjugate. This is the Wheeler propagator defined in Section 1.

The space of states generated by successive applications of a and a^+ on the symmetric vacuum has an indefinite metric.

The scalar product can be defined by means of the holomorphic representation (Faddeev and Slavnov, 1970). The functional space is formed by analytic functions $f(z)$, with the scalar product

$$\langle f, g\rangle = \int dz\, d\bar{z}\, e^{-z\bar{z}} f(z)\, \overline{g(z)} \tag{38}$$

or, in polar coordinates,

$$\langle f, g\rangle = \int_0^\infty d\rho\, \rho e^{-\rho^2} \int_0^{2\pi} d\phi\, f(z)\overline{g(z)} \tag{39}$$

The raising and lowering operators are represented by

$$a^+ = z, \qquad a = \frac{d}{dz} \tag{40}$$

The symmetric vacuum obeys

$$\left(\frac{d}{dz} z + z\frac{d}{dz}\right) f_0 = \left(1 + 2z\frac{d}{dz}\right) f_0 = 0$$

whose normalized solution is

$$f_0 = (2\pi^{3/2})^{-1/2} z^{-1/2}$$

The energy eigenfunctions are

$$f_n = \left[2\pi\Gamma\left(n + \frac{1}{2}\right)\right]^{-1/2} z^{-1/2} z^n \tag{41}$$

6. UNITARITY

In QFT, the equations of motion for the states of a system of interacting fields are formally solved by means of the evolution operator

$$U(t, t_0)|t_0\rangle = |t\rangle$$

The interactions between the quanta of the fields is supposed to take place in a limited region of space-time. The initial and final times can be taken to be $t_0 \to -\infty$ and $t \to +\infty$, thus defining the S-operator:

$$S = U(+\infty, -\infty)$$

We do not intend to discuss the possible problems of such a definition. Here we are only interested in its relation to Wheeler propagators.

Usually, the initial and final states are represented by free particles. However, when Wheeler fields are present, their quanta either mediate interactions between other particles or they end up at an absorber. This circunstance had been pointed out by Wheeler and Feynman (1945, 1949). In consequence, the S-matrix not only contains the initial and final free particles, but it also contains the states of the absorbers. Through the latter we can determine the physical quantum numbers of the Wheeler virtual "asymptotic particles." For these reasons, even if the initial and final states do not contain any Wheeler free particle, for the verification of perturbative unitarity it is necessary to take them into account.

We shall ilustrate this point with some examples. Let us consider four scalar fields $\phi_s(s = 1, \ldots, 4)$ obeying Klein–Gordon equations with mass parameters m_s^2 and the interaction $\Lambda = \lambda\phi_1\phi_2\phi_3\phi_4$. They can be written as in (30).

Unitarity implies

$$SS^+ = 1$$

or, with $S = 1 - T$,

$$T + T^+ = TT^+$$

We introduce the initial and final states and also a complete descomposition of the unit operator:

$$\langle\alpha|T + T^+|\beta\rangle = \int d\sigma_\gamma \langle\alpha|T|\gamma\rangle\langle\gamma|T^+|\beta\rangle$$

For the perturbative development

$$T = \sum_n \lambda^n T_n$$

$$\langle\alpha|T_n + T_n^+|\beta\rangle = \sum_{s=1}^{n-1} \int d\sigma_\gamma \langle\alpha|T_{n-s}|\gamma\rangle \langle\gamma|T_s^+|\beta\rangle \tag{42}$$

In particular, $T_0 = 0$ and $T_1 =$ pure imaginary.

For $n = 2$

$$\langle\alpha|T_2 + T_2^+|\beta\rangle = \int d\sigma_\gamma \langle\alpha|T_1|\gamma\rangle\langle\gamma|T_1^+|\beta\rangle \tag{43}$$

where we will take $T_1 = i\phi_1\phi_2\phi_3\phi_4$. Here ϕ_1 and ϕ_2 are supposed to be normal fields whose states can be obtained from the usual vacuum, and

$$|\alpha\rangle = a_2^+ a_1^+ |0\rangle\,, \qquad |\beta\rangle = a_{2'}^+ a_{1'}^+ |0\rangle$$

On the other hand, for ϕ_3 and ϕ_4 we leave open the possibility of a choice between the usual vacuum or the symmetric one.

The left-hand side of (43) comes from the second-order loop formed with the convolution of a propagator for ϕ_3 and another for ϕ_4. When both fields are normal, we have the convolution of two Feynman propagators, where the real part is

$$\begin{aligned}\mathrm{Re}[(P + m_3^2 - i0)^{-1} * (P + m_4^2 - i0)^{-1}]\\ = (P + m_3^2)^{-1} * (P + m_4^2) - \pi^2\delta(P + m_3^2) * \delta(P + m_4^2)\end{aligned}$$

In the physical region ($P < 0$) both terms in the r.h.s. give the same contribution:

$$\begin{aligned}\mathrm{Re}[(P + m_3^2 - i0)^{-1} * (P + m_4^2 - i0)^{-1}]\\ = 2\,(P + m_3^2)^{-1} * (P + m_4^2)^{-1} \quad (P < 0)\end{aligned} \tag{44}$$

Equation (44) implies that the left-hand side of (43) for Feynman particles is twice the value corresponding to Wheeler particles.

The relation (43) is known to be valid for normal fields, so there is no point in proving it here. We are going to show where the relative factor 2 comes from.

The decomposition of unity for normal fields is

$$\begin{aligned}\mathbf{I} &= \int d\sigma_\gamma\, |\gamma\rangle\langle\gamma| \\ &= |0\rangle\langle 0| + \int d^{\nu-1}q\; a^+(\bar{q})|0\rangle\langle 0|a(\bar{q}) \\ &\quad + \int d^{\nu-1}q_1\, d^{\nu-1}q_2 \frac{1}{\sqrt{2}}\, a^+(\bar{q}_1)a + (\bar{q}_2)|0\rangle\langle 0|a(\bar{q}_1)a(\bar{q}_2)\frac{1}{\sqrt{2}} + \cdots\end{aligned} \tag{45}$$

Then, for the T_1 matrix we have

$$\begin{aligned}\langle\alpha|T_1|\gamma\rangle = \langle\, 0|a_1(\bar{p})\phi_1(x)|0\rangle\langle 0|a_2(\bar{p})\phi_2(x)|0\rangle \\ \times \langle 0|\phi_3(x)a_3^+(\bar{q}_3)|0\rangle\langle 0|\phi_4(x)a_4^+(\bar{q}_4)|0\rangle\end{aligned} \tag{46}$$

where an integration over x space is understood.

When the fields are expressed in terms of the operators $a(q)$ and $a^{+}(q)$ as in equation (32), we obtain

$$\langle\alpha|T_1|\gamma\rangle = \frac{(2\pi)^{\nu}}{(2\pi)^{2(\nu-1)}} \frac{\delta(p - q_3 - q_4)}{4\sqrt{\omega_1\omega_2\omega_3\omega_4}} \qquad (p = p_1 + p_2) \tag{47}$$

and

$$\langle\gamma|T_1|\beta\rangle = \frac{(2\pi)^{\nu}}{(2\pi)^{2(\nu-1)}} \frac{\delta(q_3 + q_4 - p')}{4\sqrt{\omega_1'\omega_2'\omega_3\omega_4}} \qquad (p' = p_1' + p_2') \tag{48}$$

Multiplying together (47) and (48) and adding all possible $|\gamma\rangle\langle\gamma|$ (all $\bar{q}_3$ and $\bar{q}_4$), we get

$$\int d\sigma_\gamma \, \langle\alpha|T_1|\gamma\rangle\langle\gamma|T_1^{+}|\beta\rangle$$

$$= \frac{\delta(p - p')}{16(2\pi)^{2\nu-4}\sqrt{\omega_1\omega_2\omega_1'\omega_2'}} \int d\bar{q} \, \frac{\delta(p^0 - \omega_3(\bar{q}) - \omega_4(\bar{p} - \bar{q}))}{\omega_3(\bar{q})\omega_4(\bar{p} - \bar{q})} \tag{49}$$

This result coincides with (43)(l.h.s.) when the p^0-convolution is carried out.

Suppose now that one of the fields, say ϕ_4, has the Wheeler function as propagator. Instead of (44) we have

$$Re[(P + m_3^2 - i0)^{-1} * (P + m_4^2)^{-1}] = (P + m_3^2)^{-1} * (P + m_4^2)^{-1} \tag{50}$$

which is half the value of (44).

To evaluate the matrix $\langle T_1\rangle$ for this case we note that the descomposition of unity for the states of ϕ_4 (with an indefinite metric) is now

$$\begin{aligned} \mathbf{I} &= \int d\sigma_\gamma \, |\gamma\rangle\langle\gamma| \\ &= |0\rangle\langle 0| + \int d^{\nu-1}q \, \sqrt{2}a^{+}(\bar{q})|0\rangle\langle 0|a(\bar{q})\sqrt{2} \\ &\quad - \int d^{\nu-1}q \, \sqrt{2}a(\bar{q})|0\rangle\langle 0|a^{+}(\bar{q})\sqrt{2} + \cdots \end{aligned} \tag{51}$$

The normalization factors come from the VEV quoted in Section 5, equation (35). It is not necessary to evaluate again the matrix element (46). Its last vacuum expectation value has now a factor 1/2 from (35) and a factor

$\sqrt{2}$ form the normalization in (78). When the matrixes for T_1 and T_1^+ are multiplied together, we get an extra factor $(\sqrt{2}/2)^2 = 1/2$, as should be for unitarity to hold.

When both fields ϕ_3 and ϕ_4 are of the Wheeler type, we get for the convolution of the respective Wheeler propagators the same result (50).

The matrix element of T_1 gains now two factors $\sqrt{2}/2$, i.e., a factor 1/2. When we multiply $\langle T_1\rangle\langle T_1^+\rangle$ we get a factor $1/2 \cdot 1/2 = 1/4$, and we seem to be in trouble with unitarity. However, in this case a new matrix contributes to $\langle T_1\rangle$. It is

$$\begin{aligned}&\langle 0|a_1(\bar{p}_1)\phi_1(x)a_1^+(\bar{q}_1)a_1^+(\bar{q}_1')|0\rangle\\ &\times\langle 0|a_2(\bar{p}_2)\phi_2(x)a_2^+(\bar{q}_2)a_2^+(\bar{q}_2')|0\rangle\\ &\times\langle 0|\phi_3(x)a_3^+(\bar{q}_3)|0\rangle\langle 0|\phi_4(x)a_4^+(\bar{q}_4)|0\rangle\end{aligned} \tag{52}$$

(52) is only possible when both ϕ_3 and ϕ_4 are associated with symmetric vacua.

For the first matrix factor we have:

$$\begin{aligned}&\langle 0|a_1(\bar{p}_1)\phi_1(x)a_1^+(\bar{q}_1)a_1^+(\bar{q}_1')|0\rangle\\ &\quad=\delta(p_1-q_1)\frac{e^{iq_1'y}}{\sqrt{2\omega_1(q_1')}}+\delta(p_1-q_1')\frac{e^{-iq_1x}}{\sqrt{2\omega_1(q_1)}}\end{aligned} \tag{53}$$

A similar matrix factor from $\langle T_1^+\rangle$ gives

$$\begin{aligned}&\langle 0|a_1(\bar{q}_1)a_1(\bar{q}_1')\phi_1(y)a_1(\bar{p}_1')|0\rangle\\ &\quad=\delta(p_1'-q_1')\frac{e^{iq_1y}}{\sqrt{2\omega_1(q_1)}}+\delta(p_1'-q_1)\frac{e^{iq_1'y}}{\sqrt{2\omega_1(q_1')}}\end{aligned} \tag{54}$$

When we multiply together (53) and (54), the crossed terms do not contribute $[\delta(p_1-p_1')=0]$. The other two terms give equal contributions. A similar evaluation can be done for the second factor of (52) and the corresponding factor of $\langle T^+\rangle$. For this reason we are going to keep only the first terms from (53) and (54) (multiplied with the appropriate constants):

$$\begin{aligned}\langle\alpha|T_1|\gamma\rangle&=\frac{2}{(2\pi)^{2(\nu-1)}}\,\delta\,(p_1-q_1)\frac{e^{-iq_1x}}{\sqrt{2\,\omega_1(q_1)}}\,\delta\,(p_2-q_2)\\ &\quad\times\frac{e^{-iq_2x}}{\sqrt{2\omega_2(q_2)}}\frac{e^{iq_3x}}{2\sqrt{2\omega_3(q_3)}}\frac{e^{iq_4x}}{2\sqrt{2\omega_4(q_4)}}\end{aligned}$$

and after performing the x integration, we have

$$\langle\alpha|T_1|\gamma\rangle = \frac{(2\pi)^\nu}{2(2\pi)^{2(\nu-1)}} \frac{\delta(-q_1' - q_2' + q_3 + q_4)\,\delta(p_1 - q_1)\,\delta(p_2 - q_2)}{4\sqrt{\omega_1'\omega_2'\omega_3\omega_4}}$$

Analogously,

$$\langle\gamma|T_1^+|\beta\rangle = \frac{(2\pi)^\nu}{2(2\pi)^{2(\nu-1)}} \frac{\delta(q_1 + q_2 - q_3 - q_4)\,\delta(p_1' - q_1')\,\delta(p_2' - q_2')}{4\sqrt{\omega_1'\omega_2'\omega_3\omega_4}}$$

The sum $\int d\sigma_\gamma\, \langle\alpha|T_1|\gamma\rangle\langle\gamma|T_1^+|\beta\rangle$ corresponds to an integration on q_1, q_1', q_2, q_2'. It is easy to see that after this operations we get one-fourth of (49), thus completing the proof of unitarity for the proposed example.

The case in which ϕ_3 and ϕ_4 obey a KG equation with complex mass parameters can be treated in an analogous way,

Summarizing: whatever the case, any proof of unitarity for normal fields based on (42) and the decomposition of unity given by (45) can be converted into a proof of unitarity for fields with symmetric vacuum with the use of the decomposition (47).

7. DISCUSSION

We have shown that the Wheeler propagator has several interesting properties. In the first place we have the fact that it implies only virtual propagation. The on-shell δ-function, a solution of the free equation, is absent. No quantum of the field can be found in a free state. The function is always zero for spacelike distances. The field propagation takes place inside the light-cone. This is true for bradyons, but is also true for fields that obey Klein–Gordon equations with the wrong sign of the mass term and even for complex mass fields. The usual vacuum state is annihilated by the descending operator a, and gives rise to the Feynman propagator. The Wheeler Green function is associated with the symmetric vacuum. This vacuum is not annihilated by a, but rather by the "true energy" operator, a symmetric combination of annihilation and creation operators. The space of states generated by a and a^+ has an indefinite metric. There are known methods to deal with this kind of space. In particular we can define and handle all scalar products by means of the "holomorphic representation" (Faddeev and Slavnov, 1970). Due to the absence of asymptotic free waves, no Wheeler particle will appear in external legs of the Feynman diagrams. Only the propagator will appear explicitly, associated with internal lines. So the theoretical tools to deal with matrix elements in spaces with indefinite metric will not in actual fact be necessary for the evaluation of cross sections. However, the decomposition of unity for spaces with indefinite metric is needed for the proof of another important point. The inclusion of Wheeler fields and the corresponding Wheeler propagators does not produce any violation of unitarity when only normal particles are found in external legs of Feynman diagrams.

To complete the theoretical framework for a rigorous mathematical analysis, it is perhaps convenient to notice that the propagators we have defined are continuous linear functionals on the space of the entire analytic functions rapidly decreasing on the real axis. They are known in general as "tempered ultradistributions" (Sebastiao e Silva, 1958; Hasumi, 1961; Morimoto, 1978a, b; Bollini *et al.*, 1994). The Fourier-transformed space contains the usual distributions and also admits exponentially increasing functions (distributions of the exponential type) (see also Bollini *et al.*, 1996).

We must also answer the important question, What are the possible uses of the Wheeler propagators?

In the first place we would like to stress the fact that the quanta of Wheeler fields cannot appear as free particles. They can only be detected as virtual mediators of interactions. It is in the light of this observation that we must look for probable applications.

We will first take the case of a tachyon field. It is known that unitarity can not be achieved, provided we accept the implicit premise that only Feynman propagators are to be used, with the consequent presence of free tachyons. This work can also be considered to be a proof of the incompatibility of unitarity and the Feynman propagator for tachyons. Furthermore, this procedure fits naturally into the treatment for complex mass fields of Section 5. Higgs problems could be related to this case. The scalar field of the standard model behaves as a tachyon field for low amplitudes. The fact that the Higgs has not yet been observed suggests the posibility that the corresponding propagator might be a Wheeler function (Bollini and Rocca, 1997a). It is easy to see that this assumption does not spoil any of the experimental confirmations of the model (on the contrary, it adds the nonobservation of the free Higgs).

Another possible application appears in higher order equations. Those equations appear, for example, in some supersymmetric models for higher dimensional spaces (Bollini and Giambiagi, 1985). They can be decomposed into Klein–Gordon factors with general mass parmeters. The corresponding fields have Wheeler functions as propagators. It is interesting that there are models of higher order equations, coupled to electromagnetism, which can be shown to be unitary, no matter how high the order is (Bollini *et al.*, 1997).

The acceptance of tachyons as Wheeler particles might be of interest for bosonic string theory. With the symmetric vacuum we can show that the Virasoro algebra turns out to be free of anomalies in spaces of arbitrary number of dimensions (Bollini and Rocca, 1997b). The excitations of the string are Wheeler functions in this case.

ACKNOWLEDGMENT

This work was partially supported by Consejo Nacional de Investigaciones Científicas and Comisión de Investigaciones Científicas de la Pcia. de Buenos Aires, Argentina.

REFERENCES

Barci, D. G., Bollini, C. G., and Rocca, M. C. (1993). *Nuovo Cimento*, **106A**, 603.

Barci, D. G., Bollini, C. G., Oxman, L. E., and Rocca, M. C. (1994a). *International Journal of Modern Physics A*, **9**, 4169.

Barci, D. G., Bollini, C. G., and Rocca, M. C. (1994b). *International Journal of Modern Physics A*, **9**, 3497.

Bochner, S. (1939). *Lectures on Fourier Integrals*, Princeton University Press, Princeton, New Jersey, p. 235.

Bollini, C. G., and Giambiagi, J. J. (1985). *Physical Review D*, **32**, 3316.

Bollini, C. G., and Giambiagi, J. J. (1996). *Physical Review D*, **53**, 5761.

Bollini, C. G., and Oxman, L. E. (1992). *International Journal of Modern Physics A*, **7**, 6845.

Bollini, C. G., and Rocca, M. C. (1997a). *Nuovo Cimento*, **110A**, 363.

Bollini, C. G., and Rocca, M. C. (1997b). *Nuovo Cimento*, **110A**, 353.

Bollini, C. G., Oxman, L. E., and Rocca, M. C. (1994). *Journal of Mathematical Physics*, **35**, 4429.

Bollini, C. G., Civitarese, O., De Paoli, A. L., amd Rocca, M. C. (1996). *Journal of Mathematical Physics*, **37**, 4235.

Bollini, C. G., Oxman, L. E., and Rocca, M. C. (1997). *International Journal of Modern Physics A*, **12**, 2915.

Dirac, P. A. M. (1938). *Proceedings of the Royal Society of London A*, **167**, 148.

Erdelyi, A., ed., *Tables of Integral Transforms*, Vol. 2, McGraw-Hill, New York, 1954.

Faddeev, L. D., and Slavnov, A. A. (1970). *Gauge Fields. Introduction to Quantum Theory*, Benjamin Cummings.

Gradshteyn, S., and Ryzhik, I. M. (1980). *Tables of Integrals, Series, and Products*, Academic Press, New York.

Guelfand, I. M., and Shilov, G. E. (1972). *Les Distributions*, Vol. 1, Dunod, Paris.

Hasumi, M. (1961). *Tohoku Mathematical Journal*, **13**, 94.

Morimoto, M. (1978a). *Proceedings of the Japan Academy. Science*, **51**, 83.

Morimoto, M. (1978b). *Proceedings of the Japan Academy. Science*, **51**, 213.

Sebastiao e Silva, J. (1958). *Mathematische Annalen*, **136**, 38.

Wheeler, J. A., and Feynman, R. P. (1945). *Reviews of Modern Physics*, **17**, 157.

Wheeler, J. A., and Feynman, R. P. (1949). *Reviews of Modern Physics*, **21**, 425.